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On the Perturbation Expansion of the KPZ-Equation

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We present a simple argument to show that the β-function of the d-dimensional KPZ-equation ($d \geq 2$) is to all orders in perturbation theory given by

$$\beta(g_R) = (d-2)g_R - \frac{2}{(8\pi)^{d/2}}\Gamma(2-d/2)g_R^2 \ .$$

Neither the dynamical exponent z nor the roughness-exponent ζ have any correction in any order of perturbation theory. This shows that standard perturbation theory cannot attain the strong-coupling regime and in addition breaks down at $d = 4$. We also calculate a class of correlation-functions exactly.

KEY WORDS: KPZ-equation, growth processes

1 Introduction

During the last years, there has been an increasing interest in out of equilibrium dynamics. Among these, a lot of research was devoted to non-linear growth, and in particular to the Kardar-Parisi-Zhang equation [1]

$$\frac{\partial h(x,t)}{\partial t} = \nu \nabla^2 h(x,t) + \frac{\lambda}{2}\left(\nabla h(x,t)\right)^2 + \eta(x,t) \ , \tag{1.1}$$

$$\overline{\eta(x,t)\eta(x',t')} = 2D\delta^d(x-x')\delta(t-t') \ . \tag{1.2}$$

Thanks to a fluctuation dissipation theorem and the mapping to exactly solvable models, much is known for space-dimension $d = 1$ [1, 2, 3]. In contrast, the case of $d \geq 2$ can only be attacked by approximative methods or field-theoretic perturbative expansions. Using the latter, the fixed point structure of the renormalization group flow for $d = 2 + \varepsilon$ has been obtained [1, 4, 5]. Two domains can be distinguished: For small effective coupling

$$g = \frac{2\lambda^2 D}{\nu^3} \ , \tag{1.3}$$

the renormalization group flow goes to 0 in the long-wavelength limit. For large coupling the flow is expected to tend to a strong coupling fixed point $g = g_{\rm sc}$. The crossover takes place at $g = g_{\rm co}$, which turns out to be of order ε in an ε-expansion and can therefore be studied perturbatively.

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In this article we present a simple argument to resum the perturbation expansion and to calculate the renormalization group functions to all orders. This topic has first been adressed in [6], but is difficult to access there by a non-specialist. The author of the present publication was therefore encouraged to find a simple derivation, which sets the results on a clear footing and allows to study the limits of the method. Emphasis is laid upon a pedagogical presentation, understandable with an elementary background in renormalization theory. We will therefore perform all steps of the renormalization program by using elementary tools only. This includes a proof of perturbative renormalizability, which in standard field theories is a formidable task, see e.g. [7] and references therein.

Let us also mention that similar conclusions have independently been obtained by H.K. Janssen [8], and in a different context and with completely different methods in [9, 10].

2 Summation of the KPZ-equation to all orders in perturbation theory

First of all, we want to eliminate the nonlinear term in Eq. (1.1). Using the well-known Cole Hopf transformation

$$W(x,t) := \mathrm{e}^{\frac{\lambda}{2\nu}h(x,t)} \ , \tag{2.1}$$

and absorbing a factor of ν into t leads to the following equations in terms of $W(x,t)$:

$$\frac{\partial}{\partial t}W(x,t) = \Delta W(x,t) + \frac{\lambda}{2\nu^2}\eta(x,t)W(x,t) \ , \tag{2.2}$$

$$\overline{\eta(x,t)\eta(x',t')} = 2\nu D\delta^d(x-x')\delta(t-t') \ . \tag{2.3}$$

In interpreting Eqs. (1.1) and (2.2) in Itô-discretization, we have explicitly subtracted a drift term $\sim W(x,t)$. Thus the expectaton value of $W(x,t)$ will be constant.

We are now in a position to write down the generating functional for the dynamic expectation values

$$\int \mathcal{D}\left[\tilde{W}\right]\mathcal{D}\left[W\right]\mathcal{D}\left[\eta\right]\mathrm{e}^{-J\left[W,\tilde{W},\eta\right]+\int j(x,t)W(x,t)+\tilde{\jmath}(x,t)\tilde{W}(x,t)} \tag{2.4}$$

with

$$J\left[W,\tilde{W},\eta\right] = \int_{x,t}\Big[\tilde{W}(x,t)\Big(\dot{W}(x,t) - \Delta W(x,t) - \frac{\lambda}{2\nu^2}\eta(x,t)W(x,t)\Big) + \frac{1}{4D\nu}\eta^2(x,t)\Big] \ . \tag{2.5}$$

Expectation values are obtained from Eq. (2.4) through variation with respect to $j(x,t)$ and $\tilde{\jmath}(x,t)$. The interpretation of the functional (2.5) is simple: The term proportional to $\tilde{W}(x,t)$ is just the equation of motion (2.2), thus integrating over all purely imaginary fields forces the equation of motion to be satisfied. The last term in Eq. (2.5) is the noise distribution. Note that the path integral runs over positive values of $W(x,t)$ only, since $W(x,t) = \mathrm{e}^{\frac{\lambda}{2\nu}h(x,t)}$.

The noise-integration can be done. We obtain a simplified action

$$J\left[W,\tilde{W}\right]=\int_{x,t}\tilde{W}(x,t)\left(\dot{W}(x,t)-\Delta W(x,t)\right)-\frac{g}{2}\left(\tilde{W}(x,t)W(x,t)\right)^2 \ , \tag{2.6}$$

where

$$g=\frac{2\lambda^2 D}{\nu^3} \ . \tag{2.7}$$

As a side remark, let us note that another way to obtain Eq. (2.6) is to write down the generating functional for the original KPZ-equation (1.1) and then to perform a change of coordinates [8]

$$\begin{aligned} W(x,t) &:= \mathrm{e}^{\frac{\lambda}{2\nu}h(x,t)} \ , \\ \tilde{W}(x,t) &:= \tilde{h}(x,t)\,\mathrm{e}^{-\frac{\lambda}{2\nu}h(x,t)} \ . \end{aligned} \tag{2.8}$$

This transformation leaves the integration measure invariant.

Eq. (2.2) only makes sense when specifying the initial conditions, i.e. $W(x,t)$ at time $t=0$. The simplest choice $W(x,0)=0$ leads to $\partial_t W(x,0)=0$ and consequently to $W(x,t)\equiv 0$. We therefore start with

$$W(x,0)=1 \ , \tag{2.9}$$

which is equivalent to a flat initial condition for $h(x,t)$, namely $h(x,0)=0$. In order to eliminate the constant part of Eq. (2.9) from perturbation theory, we set

$$W(x,t)=1+w(x,t) \ . \tag{2.10}$$

The response-function of the non-interacting theory ("free response-function") is

$$\begin{aligned} R(x-x',t-t') &= \langle w(x,t)\tilde{W}(x',t')\rangle_0 \\ &= \Theta(t-t')\left[4\pi(t-t')\right]^{-d/2}\mathrm{e}^{-(x-x')^2/4(t-t')} \ . \end{aligned} \tag{2.11}$$

All other free expectation values vanish

$$\begin{aligned} \langle w(x,t)w(0,0)\rangle_0 &= 0 \ , \\ \langle \tilde{W}(x,t)\tilde{W}(0,0)\rangle_0 &= 0 \ . \end{aligned} \tag{2.12}$$

Let us now adress the problem of restricting the path-integral to values of $W(x,t)>0$. Starting with $W(x,t)>0$, the time evolution in Eq. (2.2) will keep $W(x,t)>0$ for all t. This is easily verified for vanishing noise, therefore the free response-function (2.11) is correct. We shall see below that also perturbation theory respects this property.

Perturbation theory is developed by starting from the functional (2.6). The non-linear term is

$$\frac{1}{2}\left(\tilde{W}(x,t)W(x,t)\right)^2=\frac{1}{2}\tilde{W}^2(x,t)+\tilde{W}^2(x,t)w(x,t)+\frac{1}{2}\left(\tilde{W}(x,t)w(x,t)\right)^2 \ , \tag{2.13}$$

and we denote

$$\begin{aligned}\frac{1}{2}\int_{x,t}\tilde{W}^2(x,t) &= \quad ,\\ \frac{1}{2}\int_{x,t}\tilde{W}^2(x,t)w(x,t) &= \quad ,\\ \frac{1}{2}\int_{x,t}\left(\tilde{W}(x,t)w(x,t)\right)^2 &= \quad .\end{aligned} \tag{2.14}$$

Since and by its own can not build up divergent diagrams, we neglect them for the moment and start by analysing the perturbative expansion of an observable O with only

$$\langle O\rangle = \langle O\,\mathrm{e}^{g}\rangle_0 \ . \tag{2.15}$$

The basic ingredient is the exponential of the interaction

$$\mathrm{e}^{g} \ , \tag{2.16}$$

from which we have to build vertices in perturbation theory. First of all, there is no vacuum-correction, as self-contractions of vanish identically due to causality. This also holds for the contraction of more than one vertex. With the same argument, we conclude, that no diagram with two external legs can be constructed. Therefore, there is no divergent contribution to both $\tilde{W}\dot{W} \equiv \tilde{W}\dot{w}$ and $\tilde{W}\Delta W \equiv \tilde{W}\Delta w$ at any order in perturbation-theory, therefore ν (hidden in t) has not to be renormalized[2]. The only possible diagrams are chains of , of the form , and so on or higher order vertices. The latter are irrelevant in perturbation theory [7].

We therefore write

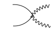
$$\begin{aligned}\mathrm{e}^{g} = 1 &+ g + g^2 \\ &+ g^3 + g^4 + \dots \\ &+ \text{higher order vertices} \ ,\end{aligned} \tag{2.17}$$

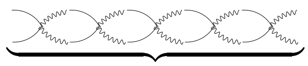
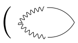
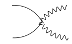
where the time-argument of the vertices grows from left to right. Note that the combinatorial factor of $\frac{1}{n!}$ which comes from the expansion of the exponential function at order g^n has canceled against the $n!$ possibilities to order the vertices in time. In addition, any bubble appears with a combinatorial factor of 2, which cancels against factors of $1/2$ from the vertex, Eq. (2.13). So any of the chain diagrams in (2.16) still contains a factor of $1/2$.

To proceed further, we first suppress the "higher order vertices" in Eq. (2.17), as the only divergencies they may contain are sub-chains as those depicted in Eq. (2.17), that will be treated here.

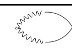
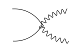
Second, we can switch to Fourier-representation, thus regard the diagrams in Eq. (2.17) as a function of the external momentum p and frequency ω instead of the coordinates x

[2]Note that this is not in contradiction with the non-trivial value for z obtained in dimension $d=1$: there, as well as for $d \geq 2$, the non-trivial fixed point describing the rough phase is in the strong coupling regime, i.e. *not* acceccible by a systematic perturbation expansion; this means that the expansion parameter is always large and the expansion uncontrolled.

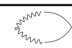
and t, and finally integrate over p and ω instead of x and t. Then, each chain in Eq. (2.17) factorizes, i.e. can be written as product of the vertex  times a power of the elementary loop diagram (which is a function of p and ω)

$\xrightarrow{p,\omega}$  $= \big($  $_{p,\omega}\big)^n$  . (2.18)

Eq. (2.17) is a geometric sum, equivalent to

$1 + g \dfrac{1}{1 - g \; \text{(loop)}_{p,\omega}}$   , (2.19)

and one reads off the effective 4-point function

$\Gamma_{ww\tilde W\tilde W}\Big|_{p,\omega} = g \dfrac{1}{1 - g \; \text{(loop)}_{p,\omega}}$  . (2.20)

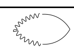
As we shall show below, the loop integral in Eq. (2.20) is divergent for any p and ω when $d \to 2$. Renormalisation means to absorb this divergence into a reparametrization of the coupling constant g: We claim that there is a function $a = a(d)$, such that the 4-point function is finite (renormalized) as a function of g_R instead of g, when setting

$$g = Z_g g_R \mu^{-\varepsilon} \tag{2.21}$$

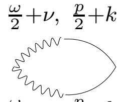
with

$$Z_g = \frac{1}{1 + a g_R}\,, \quad \varepsilon = d - 2\,. \tag{2.22}$$

μ is an arbitrary scale, the so-called renormalization scale. As a function of g_R, the 4-point function reads

$\Gamma_{ww\tilde W\tilde W}\Big|_{p,\omega} = \dfrac{g_R \mu^{-\varepsilon}}{1 + (a - \mu^{-\varepsilon} \; \text{(loop)}_{p,\omega})\, g_R}$  . (2.23)

To complete the proof, we have to calculate the elementary diagram,

$\xrightarrow{p,\omega}$  $\xrightarrow{p,\omega}$. (2.24)

This is

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d} \int \frac{\mathrm{d}\nu}{2\pi} \frac{1}{\left(\frac{p}{2}+k\right)^2 + i\left(\frac{\omega}{2}+\nu\right)} \frac{1}{\left(\frac{p}{2}-k\right)^2 + i\left(\frac{\omega}{2}-\nu\right)}\,. \tag{2.25}$$

To perform the integration over ν, the integration path can be closed either in the upper or lower half-plane. Closing it in the upper half-plane, we obtain:

$$\begin{aligned}\int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{\left(k+\frac{p}{2}\right)^2 + \left(k-\frac{p}{2}\right)^2 + i\omega} &= \int_0^\infty \mathrm{d}s \int \frac{\mathrm{d}^d k}{(2\pi)^d} \mathrm{e}^{-s\left(2k^2+\frac{1}{2}p^2+i\omega\right)} \\ &= \frac{1}{(8\pi)^{d/2}} \int_0^\infty \mathrm{d}s\, s^{-d/2} \mathrm{e}^{-s(\frac{1}{2}p^2+i\omega)} \\ &= \frac{1}{(8\pi)^{d/2}} \left(\frac{1}{2}p^2 + i\omega\right)^{d/2-1} \Gamma\left(1-\frac{d}{2}\right)\,.\end{aligned} \tag{2.26}$$

The 4-point function in Eq. (2.23) therefore depends on p and ω, and more specifically on the combination $\frac{1}{2}p^2 + i\omega$. We now chose a subtraction scale, i.e. we demand that $\Gamma_{ww\tilde{W}\tilde{W}}$ evaluated at $\mu^2 = \frac{1}{2}p^2 + i\omega$ be

$$\Gamma_{ww\tilde{W}\tilde{W}}\Big|_{\frac{1}{2}p^2+i\omega=\mu^2} = g_R \ . \tag{2.27}$$

This is achieved by setting

$$a = \frac{1}{(8\pi)^{d/2}}\Gamma\left(1-\frac{d}{2}\right) \equiv \frac{2}{(8\pi)^{d/2}}\Gamma\left(2-\frac{d}{2}\right)\frac{1}{\varepsilon} \ . \tag{2.28}$$

Moreover, since

$$\frac{1}{\varepsilon}\left(\frac{1}{2}p^2 + i\omega\right)^{d/2-1} \mu^{-\varepsilon} \tag{2.29}$$

is finite in the limit $\varepsilon \to 0$ as long as the combination of $\frac{1}{2}p^2 + i\omega$ is finite, it can be read off from Eq. (2.23) that then also $\Gamma_{ww\tilde{W}\tilde{W}}\Big|_{p,\omega}$ is finite. (If useful, either $p = 0$ or $\omega = 0$ may safely be taken.) This completes the proof. Note that this ensures that the model is renormalizable to all orders in perturbation-theory, what is normally a formidable task to show [7].

The β-function that we shall calculate now is exact to all orders in perturbation theory. As usual, it is defined as the variation of the renormalized coupling constant, keeping the bare one fixed

$$\beta(g_R) = \mu\frac{\partial}{\partial\mu}\Big|_g g_R \ . \tag{2.30}$$

From Eq. (2.27) we see that it gives the dependence of the 4-point function on p and ω for fixed bare coupling. Solving

$$g = \frac{g_R\mu^{-\varepsilon}}{1+ag_R} \tag{2.31}$$

for g_R, we obtain

$$g_R = \frac{g}{\mu^{-\varepsilon} - ag} \ , \tag{2.32}$$

and hence

$$\beta(g_R) = \varepsilon g_R(1+ag_R) \ . \tag{2.33}$$

Using a from Eq. (2.28), our final result is

$$\beta(g_R) = (d-2)g_R - \frac{2}{(8\pi)^{d/2}}\Gamma\left(2-\frac{d}{2}\right)g_R^2 \ , \tag{2.34}$$

as stated in the abstract. It shows that standard perturbation theory fails to produce a strong coupling fixed point, a result which cannot be overemphasized. This means that any treatment of the strong coupling regime has to rely on non-perturbative methods. It does of course not rule out the possibility to find an exactly solvable model, non-equivalent to KPZ, for which it is possible to expand towards the strong-coupling regime of KPZ. Note that also for $d = 1$ this equation does not possess a fixed point describing the

rough phase; the latter is in the strong coupling regime, *not* accessible by a perturbation expansion.

Let us also note that the β-function is divergent at $d = 4$, and therefore our perturbation expansion breaks down at $d = 4$. To cure the problem, a lattice regularized version of Eq. (2.2) may be used. However, then the lattice cut-off a will enter into the equations and the result is no longer model-independent. This may be interpreted as $d = 4$ being the upper critical dimension of KPZ, or as sign for a simple technical problem. Compare also [11].

3 Critical exponents and connection to calculations in the standard representation of the KPZ-equation

Quite a lot of work has been done by directly working in the KPZ-picture. A crucial point to understand is therefore the relation of the KPZ-picture and the Cole-Hopf transformed model used here. The relation is easy on the level of the β-functions, which are the same for both models. In the Cole-Hopf picture it is also easy to see that, as there are no corrections to the response function, ν and by this means the dynamical exponent z does not acquire any perturbative correction, and thus

$$z = 2 \tag{3.1}$$

to all orders.

What is not so easy to compare are correlation functions of the height-variable $h(x,t)$ in the original KPZ-language with objects of the Cole-Hopf transformed theory. One has to compare, see Eq. (2.1)

$$\langle h(x,t)\, h(x',t')\rangle \sim \langle \ln W(x,t)\, \ln W(x',t')\rangle \ . \tag{3.2}$$

As logarithms are difficult to handle, one can also study expectation values of vertex-operators

$$\left\langle \mathrm{e}^{\kappa W(x,t)} \mathrm{e}^{\rho W(x',t')} \right\rangle \tag{3.3}$$

with arbitrary κ and ρ. Expectation values of h can be reconstructed by using the identity

$$\ln W = \lim_{\delta\to 0} \frac{1}{\delta}\left[1 - W^{-\delta}\right] = \lim_{\delta\to 0} \int_0^\infty \frac{\mathrm{d}\kappa}{\kappa} \kappa^\delta \left[\mathrm{e}^{-\kappa} - \mathrm{e}^{-\kappa W}\right] \ . \tag{3.4}$$

Therefore the characteristic functions contain much more information than e.g. the 2-point function

$$\langle W(x,t)\, W(x',t')\rangle \ . \tag{3.5}$$

Using the tilt-invariance of the KPZ-model [12], Eq. (3.1) also implies that the roughness exponent ζ_h, defined via (for $\zeta_h < 0$; for $\zeta_h > 0$ one would use $\left\langle (h(x,t) - h(x',t))^2 \right\rangle$ instead)

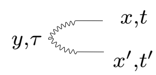
$$\langle h(x,t)\, h(x',t)\rangle \sim |x - x'|^{2\zeta_h} \tag{3.6}$$

of the h-field at a non trivial fixed point is

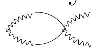
$$\zeta_h = 0 \ . \tag{3.7}$$

To study correlation-functions more directly, we use the first term from Eq. (2.14) to calculate

$$\langle w(x,t)\, w(x',t')\rangle = g \int_{y,\tau}$$



$$= g \int_{y,\tau} R(x-y,t-\tau) R(x'-y,t'-\tau) \ . \tag{3.8}$$

Note that this is the only diagram which contributes, since more complicated diagrams involving loops like , have to be taken at zero momentum and frequency and thus vanish according to Eq. (2.26).

Further, when t and t' are small, the expectation value in Eq. (3.8) is small, too. This is physical, since the surface has not much grown yet. In the other limit of large times, $t, t' \to \infty$ and keeping $t - t'$ fixed, the r.h.s. of Eq. (3.8) converges towards

$$g\, C(x-x', t-t') \ , \tag{3.9}$$

where $C(x,t)$ is the standard dynamic correlation function which in Fourier space reads

$$C(k,\omega) = \frac{1}{k^4 + \omega^2} \ . \tag{3.10}$$

For equal times, this relation reads

$$\langle w(x,t)\, w(x',t)\rangle \sim g\, |x-x'|^{2-d} = g_R Z_g |x-x'|^{2-d} \ , \tag{3.11}$$

leading to a renormalization for w of the form

$$w_R = Z_g^{-\frac{1}{2}} w \tag{3.12}$$

and to a roughness exponent ζ_w for the field w in

$$\langle w(x,t)\, w(x',t)\rangle \sim |x-x'|^{2\zeta_w} \tag{3.13}$$

with

$$\begin{aligned} \zeta_w &= \zeta_0 + \delta\zeta_w \ , \\ \zeta_0 &= \frac{2-d}{2} \ , \\ \delta\zeta_w &= \frac{1}{2} \mu \frac{\partial}{\partial \mu} \ln Z_g \ . \end{aligned} \tag{3.14}$$

Using Eqs. (2.21) and (2.22), we can express $\beta(g)$ in terms of $\delta\zeta_w(g_R)$ as

$$\beta(g_R) = g_R \left[\varepsilon - 2\delta\zeta_w\right] \ . \tag{3.15}$$

At a non-trivial fixed point $g_R = g^* > 0$, i.e. at the roughning transistion, this induces

$$\delta\zeta_w(g_R) = \frac{\varepsilon}{2} \ , \tag{3.16}$$

and therefore

$$\zeta_w = 0 \ . \tag{3.17}$$

It is tempting to indentify ζ_w with ζ_h. A priori, this is surprising, since h and w are very different observables, related by

$$w(x,t) = \mathrm{e}^{\frac{\lambda}{\nu}h(x,t)} - 1 \ . \tag{3.18}$$

The putative identity $\zeta_w = \zeta_h$ will therefore only hold, if w is, within correlation functions, well approximated by the term linear in h on the r.h.s. of Eq. (3.18). It will certainly break down when $\zeta_h > 0$, i.e. in the strong-coupling regime. It is still possible to relate correlation-functions for h and w via integral-transforms as in Eq. (3.4), as long as the expectation value in Eq. (3.3) is dominated by contractions with only, leading to purely Gaussian correlations. However, in the strong-coupling regime, also non-linear terms, of which the first is $\langle w^2(x,t)w(x',t')\rangle$, will contribute to Eq. (3.3), making a systematic analysis easier said than done. This is another difficulty of the perturbation expansion beyond the roughning transition.

On a more technical level, it is worth realizing that the above relations can be used to simplify the perturbation expansion in the original KPZ-language.

4 Conclusions

In this article, we have presented a simple method to resum the perturbation expansion of the KPZ-equation and to calculate the renormalization group β-function to all orders in perturbation theory, including a proof of perturbative renormalizability. The main conclusion is that there is no anomalous contribution to the dynamical exponent z in the weak-coupling regime and at the roughening transition. We also have given some indications of why standard perturbation theory fails in describing the strong-coupling fixed regime.

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